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Rescaled Time Domain and Heat Flux Calibration Formulation for Nonlinear Inverse Heat Conduction

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Extended Abstract: This paper describes a nonlinear surface heat flux calibration method based on rescaling principles and time-step linearization. This concept is presently developed in the context of a single in-depth temperature measurement based on linear calibration formulation approach described in Refs. 1-3. In aerospace engineering, inverse heat conduction [4] is a critical and highly practical topic as its application involves a variety of valuable and fundamental studies, such as short- and long-duration, ground- and flight-based experiments. Hypersonic flight requires reliable and predictable thermal protection systems (TPS) in order to maintain the structural integrity of the flight vehicle. Hence, establishing the suitability of materials for predefined flight scenarios is highly important. Optimal selection of TPS requires accurate predictions of the surface heat flux and temperature that accounts for thermophysical property variations as a function of temperature.

In the context of a linear framework, a transformative calibration methodology with experimental verification has been proposed at the University of Tennessee, Knoxville that provides accurate resolution of the surface heat flux. This method relates the net unknown surface heat flux to a calibration run surface heat flux and the corresponding in-depth temperature measurements during the calibration and run stages. The resulting inverse statement is expressed in terms of a Volterra integral equation of the first kind for unknown surface heat flux. It possesses a physical meaning in terms of input-output variables (input: the known surface net heat flux at the calibration stage; and, output: measured temperatures at both the calibration and run stages). Due to ill-posed nature of this inverse problem, it is necessary to stabilize the resulting Volterra integral equation of the first kind through regularization. In this process, determining the optimal regularization parameter that provides a stable and accurate prediction without over-smoothing the result is sought. That is, the predicted result should keep a balance between excess high frequency noise and insufficient low frequency signal. For the present formalism, a good regularization can be introduced

through either local-future information approach [1-3,5] and/or singular-value decomposition [6,7].

The nonlinear governing equation for one-dimensional conduction heat transfer in a slab with an adiabatic back surface at x=*L* is given by [8]

$$\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right]=\rho c_p(T)\frac{\partial T}{\partial t} \qquad x\in(0,L) \quad t\geq 0 \tag{1a}$$

subject to boundary and initial conditions of

$$\begin{aligned} q_s''(t)&=q''(0,t)=-k(T)\frac{\partial T}{\partial x}(0,t) \\ \frac{\partial T}{\partial x}(L,t)&=0 \\ T(x,0)&=T_o \qquad x\in[0,L] \end{aligned} \tag{1b-d}$$

The calibration integral formulation proposed in Refs. 1-3 for constant thermophysical properties applicable to this geometry is

$$\int_{u=0}^{t} q_r''(u)T_c(b,t-u)\mathrm{d}u=\int_{u=0}^{t} q_c''(u)T_r(b,t-u)\mathrm{d}u \qquad t\geq 0 \tag{1e}$$

where T_c is the difference between the measured calibration temperature at some depth b>0 and the initial temperature T_o; q_c" is the net surface heat flux imposed during the calibration test; T_r is the difference between the measured temperature of the same thermocouple in response to an unknown heat flux and its initial temperature T_o; and q_r" is the unknown surface heat flux to be predicted. This calibration integral equation has broad appeal as it is valid for the semi-infinite geometry as well as a finite slab where the back surface is subjected to the same constant heat transfer coefficient between the two runs.

The linear one-probe calibration model has been experimentally verified [3] to a good accuracy for a maximum slab temperature of 100^{o}C. However, in hypersonic flight, a large temperature variation is expected due to aerodynamic heating effects. Under flight conditions, the variable property effects may be significant depending on the material and temperature range such that a linear model will no longer be applicable. In this paper, a piecewise (time-step linearization) assumption is proposed to extend the linear approach given in Eq. (1e) to a new rescaled time domain and heat flux calibration formulation that is applicable to the heat equations given in Eq. (1a-d) even though the nonlinearity dominates.

The proposed approach involves a whole time domain discretization including a successive series of small time steps, Δt, in any of which all thermal properties are assumed to be constant throughout the spatial domain but differ between the time steps. Through this simplification, the nonlinear governing equation for the one-dimensional problem with an adiabatic back surface could be expressed equivalently in a series of linear governing equations whose thermophysical properties have been evaluated at their respective small time step Δt using the measured temperature at that time step.

Because the local temperature magnitudes have a pronounced variation among different time steps, it is necessary to rescale the time domain and heat flux magnitude based on the precise thermal property functions such that all the piecewise governing equations and boundary conditions keep consistency for each time step. The rescaled in-depth temperature is then considered as a linear thermal response from a rescaled surface heat flux. Thus, the linear calibration method as defined in Eq. (1e) becomes valid. To review, the major assumption for this approach is that at every moment, the distribution of thermophysical properties in spatial domain could be considered as uniform or close to uniform. It is interesting to note that this formulation still provides excellent accuracy for some materials with low thermal diffusivity (stainless steel) where a large temperature difference in the spatial domain may be observed.

Consider the problem in Eq. (1a-d) with the assumption of piecewise time-step linearization. In any small time step from $i\Delta t$ to $(i+1)\Delta t$, the governing equation can be rewritten as

$$\alpha(T(b,i\Delta t))\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial t} \qquad x\in(0,L) \quad t\in[i\Delta t,(i+1)\Delta t] \tag{2a}$$

subject to boundary and initial conditions of

$$\begin{aligned} &q''_s(t)=q''(0,t)=-k(T(b,i\Delta t))\frac{\partial T}{\partial x}(0,t)\\ &\frac{\partial T}{\partial x}(L,t)=0\\ &T(x,i\Delta t)=T(x,i\Delta t) \qquad x\in[0,L] \end{aligned} \tag{2b-d}$$

Let $(t\text{-}i\Delta t)=n_i u$ and multiply both sides of boundary condition at x=0 by the constant m_i. Doing so, we can recast the original system into the rescaled and transformed system given as

$$\begin{aligned} &\alpha(T(b,i\Delta t))\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial (n_i u)} \qquad n_i u\in[0,\Delta t] \quad x\in(0,L)\\ &m_i q''(0,n_i u+\Delta t)=-m_i k(T(b,i\Delta t))\frac{\partial T}{\partial x}(0,n_i u+\Delta t)\\ &\frac{\partial T}{\partial x}(L,n_i u+\Delta t)=0\\ &T(x,n_i u+i\Delta t=i\Delta t)=T(t=i\Delta t) \end{aligned} \tag{3a-d}$$

Next we define

$$\begin{aligned} &T_i(x,u)=T(x,n_i u+\Delta t)\\ &q''_i(u)=q''(0,n_i u+\Delta t)\\ &m_i=k(T(b,0))/k(T(b,i\Delta t))\\ &n_i=\alpha(T(b,0))/\alpha(T(b,i\Delta t)) \end{aligned} \tag{4a-d}$$

Using the above definitions, Eq. (3a-d) can be simplified to

$$\begin{aligned} &\alpha\left(T(b,0)\right)\frac{\partial^2 T_i}{\partial x^2}=\frac{\partial T_i}{\partial u} \qquad u\in\left[0,\Delta t/n_i\right] \quad x\in(0,L) \\ &m_i q_i''(u)=-k\left(T(b,0)\right)\frac{\partial T_i}{\partial x}(0,u) \\ &\frac{\partial T_i}{\partial x}(L,u)=0 \\ &T_i(x,u=0)=T(x,t=i\Delta t) \end{aligned} \tag{5a-d}$$

It is easy to observe that all the thermophysical properties in Eq. (5a-d) are constant and independent of the index of time step *i*. As a result, if all the temperatures (T_i) from time zero (i=0) to final time (i=N) are collected in sequence then it becomes the linear thermal response from the heat flux following the reconstitution of $m_i q_i$" from i=0 to i=N.

To obtain a final form of the new rescaled time domain and heat flux calibration equation, we define

$$\begin{aligned} &\bar{t}=\int_{u=0}^{t}\frac{\alpha\left(T(b,u)\right)}{\alpha\left(T(b,0)\right)}\mathrm{d}u \\ &\bar{q}''(\bar{t})=\frac{k\left(T(b,0)\right)}{k\left(T(b,t)\right)}q''(t) \\ &\bar{T}(x,\bar{t})=T(x,t) \end{aligned} \tag{6a-c}$$

The resulting Volterra integral equation of the first kind becomes:

$$\int_{u=0}^{\bar{t}}\bar{q}_r''(0,u)\bar{T}_c(b,\bar{t}-u)\mathrm{d}u=\int_{u=0}^{\bar{t}}\bar{q}_c''(0,u)\bar{T}_r(b,\bar{t}-u)\mathrm{d}u \qquad t\geq 0 \tag{6d}$$

Once the unknown heat flux in rescaled time domain has been predicted, it is necessary to map it back to the normal time domain depending on following rule

$$q''(t)=\frac{k\left(\bar{T}(b,\bar{t})\right)}{k\left(\bar{T}(b,0)\right)}\bar{q}(\bar{t}) \tag{6e}$$

Note, that the temperature $T(b,t)$ used in (6a-c) is the difference between the in-depth temperature at b>0 and the initial temperature T_o. This nonlinear calibration integral equation also works when the semi-infinite assumption is adopted or the back surface is subjected to the same constant heat transfer coefficient during the two runs.

This paper will 1) present the algorithm for above description in a concise and clear manner using visual aids for describing the mathematical manipulations and intrinsic assumptions, 2) describe the regularization method used for constructing the stable heat flux predictions, 3) present numerical results based on simulated data obtained from a highly accurate finite-control volume solution to the forward problem with prescribed heat fluxes, and 4) discuss the extension of the approach to problems with two unknown boundary conditions.

References:

1. J.I. Frankel and M. Keyhani, *Shock Waves*, online available, (2012, in press).
2. J.I. Frankel, M. Keyhani, M., and B.E. Elkins, *J. Thermophysics and Heat Transfer*, (in press)
3. B.E., Elkins, M. Keyhani, M., and J.I. Frankel, *J. Thermophysics and Heat Transfer*, (in press).
4. J.V. Beck, B. Blackwell, and C.R. St. Clair, Jr., *Inverse Heat Conduction*, Wiley, NY,1985.
5. P. Lamm, Eds.: D. Colton, H. W. Engl, A. Louis, J. R. McLaughlin, W. Rundell, Springer (Vienna, New York), 53-82, (2000).
6. J.R.Shenefelt, R Luck, R.P.Taylor. J.T.Berry, *International Journal of Heat Transfer and Mass Transfer*, 45 (2002) 67-74.
7. J.A.M Garcia, *International Journal of Thermal Sciences*, 48 (2009) 1083-1093.
8. M.N. Ozisik, Heat Conduction, Wiley, NY 1980.